# Math 247A Lecture 9 Notes

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January 27, 2020

## 1 Boundedness Properties of The Hardy-Littlewood Maximal Function and $A_p$ Weights

### 1.1 Boundedness properties of the Hardy-Littlewood maximal function

The Hardy-Littlewood maximal function is given by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y|\, dy.$$

**Theorem 1.1.** *Let $\omega : \mathbb{R}^d \to [0,\infty)$ be a locally integrable function (a weight), to which we associate a measure via*

$$\omega(E) = \int_E \omega(x)\, dx.$$

*Then*

1. *$M : L^1(M\omega\, dx) \to L^{1,\infty}(\omega\, dx)$ maps boundedly; that is,*

$$\omega(\{x : Mf(x) > \lambda\}) \lesssim \frac{1}{\lambda} \int |f(y)|(M\omega)(y)\, dy$$

*uniformly in $\lambda > 0$ for all $f \in L^1(M\omega\, dx)$.*

2. *$M : L^p(M\omega\, dx) \to L^p(\omega\, dx)$ boundedly for all $1 < p \leq \infty$; that is,*

$$\int |Mf(x)|^p \omega(x)\, dx \lesssim \int |f(y)|^p (M\omega)(y)\, dy$$

*uniformly for $f \in L^p(M\omega\, dx)$.*

Just like the proof of the maximal inequality, we will start with a covering lemma.

**Lemma 1.1** (Vitali)**.** *Given a finite collection of balls $\{B(x_j, r_j)\}_{j\in J}$, there exists a subcollection $S$ such that*

1. *Distinct balls are disjoint.*
2. $\bigcup_{j\in I} B(x_j, r_j) \subseteq \bigcup_{j\in S} B(x_j, 3r_j)$.

*Proof.* We run the following algorithm. Set $S = \varnothing$.

1. Choose a ball of largest radius and add it to $S$.
2. Discard any balls that intersect balls in $S$.
3. If no balls remain, stop. Otherwise, return to step 1. $\square$

Now let's prove the theorem.

*Proof.* First note that $M : L^\infty(M\omega\, dx) \to L^\infty(\omega\, dx)$ boundedly:

$$\|Mf\|_{L^\infty(\omega\, dx)} = \inf_{E:\omega(E)=0} \sup_{x\in E^c} Mf(x)$$

Since $\omega$ is locally integrable, it takes Lebesgue-null sets to $\omega$-null sets.

$$\begin{aligned} &\le \inf_{E:|E|=0} \sup_{x\in E^c} Mf(x) \\ &\le \|f\|_{L^\infty(dx)} \\ &= \inf_{E:|E|=0} \sup_{x\in E^c} |f(x)| \end{aligned}$$

$M\omega > 0$ unless $\omega \equiv 0$, so

$$\begin{aligned} &= \inf_{E:(M\omega)(E)=0} \sup_{x\in E^c} |f(x)| \\ &= \|f\|_{L^\infty(M\omega\, dx)}. \end{aligned}$$

So by the Marcinkiewicz interpolation theorem, it suffices to prove $M : L^1(M\omega\, dx) \o L^{1,\infty}(\omega\, dx)$.

Fix $\lambda > 0$. Let $K$ be a compact subset of $\{x : Mf(x) > \lambda\}$ (this suffices by regularity). For $x \in K$, there is some $r(x) > 0$ such that

$$\frac{1}{|B(x, r(x))|} \int_{B(x,r(x))} |f(y)|\, dy > \lambda.$$

Now $K \subseteq \bigcup_{x\in K} B(x, r(x))$, and by compactness, there exists a finite subcover such that $\bigcup_{j\in J} B(x_j, r_j)$. By Vitali, there exists a subcollection $S$ of pairwise disjoint balls such that $K \subseteq \bigcup_{j\in S} B(x_j, 3r_j)$. So $\omega(K) \le \sum_{j\in S} \omega(B(x_j, 3r_j))$.

For Lebesgue measure, we would just pull out the constant 3 and add the measures. But here, we don't have that property, so we will relate it to the maximal function. For $x \in B(x_j, r_j)$,

$$\omega(B(x_j, 3r_j)) = \int_{B(x_j,3r_j)} \omega(y)\, dy$$

$$\leq \frac{|B(x,4r_j)|}{|B(x,4r_j)|}\int_{B(x,4r_j)} \omega(y)\,dy$$
$$\leq 4^d|B(x,4r_j)|M\omega(x).$$

Now integrate this against $f$:

$$\omega(B(x_j,3r_j))\frac{1}{|B(x_j,r_j)|}\int_{B(x_j,r_j)} |f(y)|\,dy \leq 4^d \int_{B(x_j,r_j)} M\omega(x)|f(y)|\,dy. \qquad \square$$

**Remark 1.1.** Rather than placing the weights outside the maximal function, one could place them inside: Define

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f(y)|\,d\mu(y),$$

where $\mu$ is a nonnegative measure. If $\mu$ is a **doubling measure** (i.e. if $B_1 = B(x,r)$ and $B_2 = B(x,2r)$, then $\mu(B_2) \lesssim \mu(B_1)$ uniformly for $x \in \mathbb{R}^d$ and $r > 0$), then with small modifications, the proof of this theorem yields:

$$M_\mu : L^1(d\mu) \to L^{1,\infty}(d\mu), \qquad M_\mu : L^p(d\mu) \to L^p(d\mu), \qquad \forall 1 < p \leq \infty$$

boundedly.

## 1.2 $A_p$ weights

Can one characterize the nonnegative measure $\mu$ for which

$$M : L^p(d\mu) \to L^p(d\mu), \qquad 1 < p < \infty$$

boundedly? Yes, these are the $A_p$ weights.

**Definition 1.1.** We say that a locally integrable weight $\omega : \mathbb{R}^d \to [0,\infty)$ satisfies the $A_1$ condition (and we write $\omega \in A_1$) if there is a $C > 0$ such that $M\omega(x) \leq C\omega(x)$ for almost every $x$.

**Remark 1.2.** If $\omega \in A_1$, then the theorem yields

$$M : L^p(\omega\,dx) \to L^p(\omega\,dx) \qquad M : L^1(\omega, dx) \to L^{1,\infty}(\omega\,dx) \qquad \forall 1 < p \leq \infty$$

boundedly.

Let's characterize these weights.

**Lemma 1.2.** *The following are equivalent:*

*1.* $\omega \in A_1$

*2.*

$$\frac{1}{|B|}\int_B \omega(y)\,dy \lesssim \omega(x)$$

*uniformly for a.e. $x \in B$ and all balls $B$.*

*3.*

$$\frac{1}{|B|}\int_B f(y)\,dy \lesssim \frac{1}{\omega(B)}\int_B f(y)\omega(y)\,dy$$

*for all balls $B$ and all $f \geq 0$.*

*Proof.* (1) $\implies$ (2): Fix $x$ with $M\omega(x) \leq C\omega(x)$, and let $B$ be a ball of radius $r$ that contains $x$. Then

$$\begin{aligned}\frac{1}{|B|}\int_B \omega(y)\,dy &\leq \frac{2^d}{|B(x,2r)|}\int_{B(x,2r)} \omega(y)\,dy \\ &\leq 2^d M\omega(x) \\ &\leq 2^d C\omega(x).\end{aligned}$$

(2) $\implies$ (3): $\omega$ is bounded below by its maximal function, so

$$\begin{aligned}\frac{1}{\omega(B)}\int_B f(y)\omega(y)\,dy &\geq \frac{1}{\omega(B)}\int_B f(y)\left(\frac{1}{|B|}\int_B \omega(z)\,dz\,dy\right) \\ &\geq \frac{1}{|B|}\int_B f(y)\,dy.\end{aligned}$$

(3) $\implies$ (2): Let $x$ be a Lebesgue point for $\omega$, and let $B \ni x$. Let $r \ll 1$ be such that $B(x,r) \subseteq B$. Set $f = \mathbb{1}_{B(x,r)}$. Then

$$\frac{1}{|B|}|B(x,r)| \lesssim \frac{1}{\omega(B)}\int_{B(x,r)} \omega(y)\,dy.$$

Rearranging this, we get

$$\frac{\omega(B)}{|B|} \lesssim \frac{1}{|B(x,r)|}\int_{B(x,r)} \omega(y)\,dy \to \omega(x). \qquad \square$$

**Definition 1.2.** We say that a weight $\omega : \mathbb{R}^d \to [0,\infty)$ satisfies the $A_p$ condition for $1 < p < \infty$ if there exists an $A > 0$ such that

$$\sup_{\text{balls } B} \frac{1}{|B|}\int_B \omega(y)\,dy \cdot \left[\frac{1}{|B|}\int_B \omega(y)^{-p'/p}\,dy\right]^{p/p'} \leq A,$$

or equivalently,

$$\sup_{\text{balls } B} |B|^{-p}\omega(B)\|\omega^{-1/(p-1)}\|_{L^1(B)}^{p-1} \leq A.$$

**Remark 1.3.**

1. This condition is invariant under $\omega \mapsto \lambda\omega$ and $\omega(x) \mapsto \omega(\lambda x)$.

2. $\omega \in A_p$ if and only if $\sigma = \omega^{-p'/p} \in A_{p'}$. Indeed, the condition reads:

$$\sup_{\text{balls } B} \frac{1}{|B|^p} \int \sigma(y)^{-p/p'}\, dy \left[\int_B \sigma(y)\, dy\right]^{p/p'} \leq A.$$

If we raise everything to the power $p'/p$,

$$\sup_{\text{balls } B} \frac{1}{|B|^{p'}} \int_B \sigma(y)\, dy \left[\int_B \sigma(y)^{-p/p'}\, dy\right]^{p'/p} \leq A^{p'/p}.$$